HERMITE EXPANSIONS AND HARDY'S THEOREM

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Abstract. Assuming that both a function and its Fourier transform are dominated by a Gaussian of large variance, it is shown that the Hermite coefficients of the function decay exponentially. A sharp estimate for the rate of exponential decay is obtained in terms of the variance, and in the limiting case (when the variance becomes so small that the Gaussian is its own Fourier transform), Hardy's theorem on Fourier transform pairs is obtained. A quantitative result on the confinement of particle-like states of a quantum harmonic oscillator is obtained. A stronger form of the result is conjectured. Further, it is shown how Hardy's theorem may be derived from a weak version of confinement without using complex analysis.

1. Introduction

If $f \in L^1(\mathbb{R})$, the *Fourier transform* of f is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-i\xi x}\, dx.$$

Let $g_a(x) = e^{-ax^2/2}$. Hardy's theorem is usually stated as follows (see [4, Theorem 7.6], where the notation is slightly different).

Theorem 1.1. *For $a > 0$, let*

$$E(a) = \{f \in L^1(\mathbb{R}) \mid |f(x)| \le C g_a(x) \text{ and } \left|\hat{f}(\xi)\right| \le C g_a(\xi) \quad \text{for some } C \in \mathbb{R}\}.$$

If $a > 1$ then $E(a) = 0$. If $a = 1$ then $E(a) = \mathbb{C} g_a$. If $a < 1$ then $\dim E(a) = \infty$.

The last part of the trichotomy is usually substantiated by showing that all Hermite functions belong to $E(a)$, if $a < 1$.

This statement of Hardy's theorem appears to suggest that if $a < 1$ then no significant restriction is placed on f. This is far from the truth. In fact, regardless of the value of a, elements of $E(a)$ may be characterized by the rate of exponential decay of their Hermite coefficients.

Hardy's theorem is usually proved by applying the Phragmen-Lindelöf principle to the Fourier-Laplace transform of f. Instead, we apply the Phragmen-Lindelöf principle to the

Bargmann transform (the unitary intertwiner between the Schrödinger and Fock realizations of the canonical representation of the Heisenberg group). This transform is better suited for studying Hermite expansions.

The result on exponential decay of Hermite coefficients leads, via Mehler's formula to a Gaussian bound on the solutions of the Schrödinger equation for the harmonic oscillator Hamiltonian, when the initial data belong to $E(a)$. We refer to this result as *confinement*. We state a stronger conjecture.

The solutions of the harmonic oscillator Schrödinger equation are orbits of the standard maximal compact subgroup $K = \mathrm{SO}(2)$ of $\mathrm{SL}(2, \mathbb{R})$ under the metaplectic representation. Further, the K-types are precisely the Hermite functions. Using this idea, we show that Hardy's theorem follows from a weak version of confinement. Thus, if a weak confinement result is proved by purely PDE methods, we would have a proof of Hardy's theorem that does not use complex analysis. This would answer a question of Sundari.

Others have considered the connection between Hardy's theorem and Schrödinger equations. Chanillo [1] showed that Hardy's theorem is equivalent to a uniqueness theorem for the free-particle Schrödinger equation. The free-particle flow is the orbit of a unipotent subgroup of $\mathrm{SL}(2, \mathbb{R})$ under the metaplectic representation. It would be interesting to understand the connection between Hardy's theorem and the metaplectic representation better; perhaps there is a purely representation theoretic proof of Hardy's theorem!

For more on the connections between analysis and the metaplectic representation, see Howe [5] or Folland [3]. For more on Hardy's theorem, see Thangavelu [6]. Information on Hermite functions and Mehler's formula may also be found in [3, 6].

In this work, we will use the measure $dm = dx/\sqrt{2\pi}$ to define the norm on $L^p(\mathbb{R})$.

2. Exponential decay of Hermite coefficients

We will use some properties of the Bargmann transform (see [2, p78], where there seems to be a normalization error) in the proof of the main theorem. To avoid cluttering up the main argument, we recall these first.

Let $\mathcal{H}$ denote the Hilbert space of all entire functions F on $\mathbb{C}$ such that

$$\|F\|^2 = \int |F(w)|^2 \, \frac{e^{-|w|^2/2} \, du \, dv}{2\pi} < \infty \quad (w = u + iv).$$

Define $U : L^2(\mathbb{R}) \to \mathcal{H}$ by

$$Uf(w) = \frac{e^{-w^2/4}}{2^{1/4}\pi^{1/2}} \int e^{xw} e^{-x^2/2} f(x) \, dx.$$

Then Uf is defined for Schwartz class functions f, and extends to an isometric isomorphism. We call U the *Bargmann transform.* Note that

$$(U\hat{f})(w) = Uf(-iw),$$

for all $w \in \mathbb{C}$. Further, if φ_k denotes the k-th normalized Hermite function, then

$$U\varphi_k(w) = \frac{w^k}{\sqrt{2^k k!}}.$$

Theorem 2.1. *Let $a \in (0,1)$. If*

$$|f(x)| \le C g_a(x) \text{ and } \left|\hat{f}(\xi)\right| \le C g_a(\xi)$$

then

$$|\langle f, \varphi_k\rangle| \le C\sqrt{\frac{2\pi k!}{1+a}}(e/k)^{k/2}\left(\frac{1-a}{1+a}\right)^{k/4}$$

for $k = 1, 2, \ldots$.

Proof. Write $w = u + iv = re^{i\theta}$. From the first hypothesis, we obtain

$$\begin{aligned}\left|\int e^{xw}e^{-x^2/2}f(x)\,dx\right| &\le C\int e^{xu-(1+a)x^2/2}\,dx \\ &= Ce^{\frac{u^2}{2(1+a)}}\int e^{-\frac{1+a}{2}\left(x-\frac{u}{1+a}\right)^2}\,dx \\ &= C\sqrt{\frac{2\pi}{1+a}}e^{\frac{u^2}{2(1+a)}}.\end{aligned}$$

Therefore,

$$\begin{aligned}|Uf(w)| &\le C\sqrt{\frac{2\pi}{1+a}}\exp\left(\frac{v^2-u^2}{4}+\frac{u^2}{2(1+a)}\right) \\ &= C\sqrt{\frac{2\pi}{1+a}}\exp\frac{v^2+\mu u^2}{4} \\ &= C\sqrt{\frac{2\pi}{1+a}}\exp\frac{(\mu+(1-\mu)\sin^2\theta)r^2}{4},\end{aligned}$$

where $\mu = \frac{1-a}{1+a}$.

From the second hypothesis and the previous calculation, we obtain

$$\begin{aligned}|Uf(w)| &= \left|U\hat{f}(iw)\right| \\ &\le C\sqrt{\frac{2\pi}{1+a}}\exp\frac{(\mu+(1-\mu)\sin^2(\theta+\pi/2))r^2}{4} \\ &= C\sqrt{\frac{2\pi}{1+a}}\exp\frac{(\mu+(1-\mu)\cos^2\theta)r^2}{4}.\end{aligned}$$

A substantial improvement in these estimates may be obtained by applying the Phragmen-Lindelöf principle to the *holomorphic* function Uf. Let $\theta_0 = \frac{1}{2}\arctan\left(\frac{2\sqrt{\mu}}{1-\mu}\right)$, $\theta_1 = \frac{\pi}{2} - \theta_0$. Observe that $\theta_1 - \theta_0 < \frac{\pi}{2}$. Let

$$F(w) = \exp\left(i\frac{\sqrt{\mu}}{4}w^2\right)Uf(w).$$

Then F is entire, bounded by $3Ce^{|w|^2}$ everywhere, and by $C\sqrt{\frac{2\pi}{1+a}}$ on the rays $\theta = \theta_0$ and $\theta = \theta_1$. It follows from the Phragmen-Lindelöf principle that

$$|F(w)| \le C\sqrt{\frac{2\pi}{1+a}}$$

for $\theta_0 \le \theta \le \theta_1$. So

$$|Uf(w)| \le C\sqrt{\frac{2\pi}{1+a}}\exp\left(\frac{\sqrt{\mu}\sin 2\theta}{4}r^2\right) \tag{1}$$

for $\theta_0 \le \theta \le \theta_1$. Combining this with the previous two estimates, we obtain a crude estimate for Uf in the first quadrant:

$$|Uf(w)| \le C\sqrt{\frac{2\pi}{1+a}}\exp\left(\frac{\sqrt{\mu}}{4}r^2\right).$$

The same argument works in the other three quadrants, and so the estimate holds everywhere.

If $Uf(w) = \sum_{n=1}^{\infty} c_n w^n$, the Cauchy estimates give

$$|c_n| \le C\sqrt{\frac{2\pi}{1+a}}\exp\left(\frac{\sqrt{\mu}}{4}r^2\right)r^{-n}$$

for all $r > 0$. Optimizing with respect to r, we get

$$|c_n| \le C\sqrt{\frac{2\pi}{1+a}}\left(\frac{e\sqrt{\mu}}{2n}\right)^{n/2}.$$

Therefore

$$
\begin{aligned}
|\langle f, \varphi_k \rangle| &= |\langle Uf, U\varphi_k \rangle| \\
&= \int\int \left(\sum_{n=0}^{\infty} c_n w^n \right) \overline{\left(\frac{w^k}{\sqrt{2^k k!}} \right)} \frac{e^{-r^2/2}\, du\, dv}{2\pi} \\
&= \frac{|c_k|}{\sqrt{2^k k!}} \int\int r^{2k} \frac{e^{-r^2/2}\, du\, dv}{2\pi} \\
&= \sqrt{2^k k!}\, |c_k| \\
&\leq C \sqrt{\frac{2\pi k!}{1+a}} (e/k)^{k/2} \mu^{k/4}
\end{aligned}
$$

□

If $f \in E(1)$, then there exists a constant C such that

$$
|f(x)| \leq C g_a(x) \text{ and } \left| \hat{f}(\xi) \right| \leq C g_a(\xi)
$$

for all $a \in (0,1)$. So for $k \geq 1$ we have

$$
|\langle f, \varphi_k \rangle| \leq C \sqrt{\frac{2\pi k!}{1+a}} (e/k)^{k/2} \mu^{k/4},
$$

for all $\mu \in (0,1)$. It follows that $\langle f, \varphi_k \rangle = 0$ for $k \geq 1$, and so $f \in \mathbb{C}\varphi_0$. If $a > 1$ and $f \in E(a)$, then in particular $f \in E(1)$, and so $f = C\varphi_0$. However, $\varphi_0 \notin E(a)$, so $C = 0$ and $f = 0$. So the classical Hardy theorem follows from Theorem 2.1.

If $a \in (0,1)$ and $f \in E(a)$ then

$$
\langle f, \varphi_k \rangle = O\left(k^{1/4} \left(\frac{1-a}{1+a} \right)^{k/4} \right)
$$

by Theorem 2.1 and the bound $k! \leq 3\sqrt{k}(k/e)^k$, $k = 1, 2, \ldots$. In particular, if $a \in (0,1)$, $f \in E(a)$ and $\tanh(2\alpha) < a$ then

$$
\langle f, \varphi_k \rangle = O(e^{-\alpha k}). \tag{2}
$$

To obtain the endpoint estimate ($\tanh(2\alpha) = a$), we need to use the full strength of the estimate (1).

Theorem 2.2. *If $f \in E(\tanh(2\alpha))$, then*

$$
\langle f, \varphi_k \rangle = O(e^{-\alpha k}).
$$

Proof. We will use the notation from the proof of Theorem 2.1 with $a = \tanh 2\alpha$. So $\mu = e^{-4\alpha}$. Assume that f has norm at most 1. Define

$$r_n(t) = \begin{cases} \sqrt{\frac{2n+2}{\mu+(1-\mu)\sin^2 t}}, & 0 \le t < \theta_0 \\ \sqrt{\frac{2n+2}{\sqrt{\mu}\sin 2t}}, & \theta_0 \le t \le \frac{\pi}{4}. \end{cases}$$

Extend r_n to $[0, \pi/2]$ by the rule

$$r_n(t) = r_n(\frac{\pi}{2} - t), \quad \frac{\pi}{4} < t \le \frac{\pi}{2},$$

and to $[0, 2\pi]$ by $(\pi/2)$-periodicity. Then r_n is positive, continuous and piecewise smooth. Put $\gamma_n(t) = r_n(t)e^{it}$. Then each γ_n winds once about the point $w = 0$. By the Cauchy integral formula, the estimate (1) and the eightfold symmetry,

$$\begin{aligned} |c_n| &\le \frac{1}{2\pi} \int_{\gamma_n} |(Uf)(w)| \; |w|^{-(n+1)} \; |dw| \\ &= \frac{4}{\pi} \sqrt{\frac{2\pi}{1+a}} \exp\left(\frac{n+1}{2}\right) (2n+2)^{-n/2} (I_n + J_n), \end{aligned}$$

where

$$I_n = \int_0^{\theta_0} (\mu + (1-\mu)\sin^2 t)^{\frac{n-2}{2}} \sqrt{\mu^2 + (1-\mu^2)\sin^2 t}\, dt$$

and

$$J_n = \mu^{n/4} \int_{\theta_0}^{\pi/4} (\sin 2t)^{\frac{n-2}{2}}\, dt.$$

We estimate

$$\begin{aligned} I_n &\le \int_0^{\theta_0} \left(\frac{2\mu}{1+\mu}\right)^{\frac{n-2}{2}} \sqrt{\mu}\, dt \\ &= \theta_0 \frac{1+\mu}{2\sqrt{\mu}} \left(\frac{2\mu}{1+\mu}\right)^{n/2}, \end{aligned}$$

and

$$\begin{aligned} J_n &\le \mu^{n/4} \int_0^{\pi/4} (\sin 2t)^{\frac{n-2}{2}}\, dt \\ &= \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{n}{4}\right)}{\Gamma\left(\frac{n+2}{4}\right)} \mu^{n/4} \\ &\le \frac{\sqrt{6\pi}}{4} n^{-1/2} \mu^{n/4}. \end{aligned}$$

Since $(2\mu)/(1+\mu) < \sqrt{\mu}$, it follows that $I_n = o(J_n)$, and so

$$c_n = O\left(2^{-n/2}(e/n)^{n/2} n^{-1/2} \mu^{n/4}\right).$$

It follows, as before, that

$$\langle f, \varphi_k \rangle = O\left(k^{-1/4}\mu^{k/4}\right) = O(e^{-\alpha k}).$$

$\square$

Example 2.3. *With $a = \tanh(2\alpha)$, let*

$$f(x) = \exp\left(\frac{-a + i\sqrt{1-a^2}}{2}x^2\right).$$

Then $f \in E(\tanh 2\alpha)$, but for all $\beta > 1$ there exists $c_\beta > 0$ such that

$$|\langle f, \varphi_k \rangle| \geq c_\beta k^{-\beta/4} e^{-\alpha k}, \quad k = 2, 4, 6, \ldots.$$

So Theorem 2.2 is sharp.

3. Confinement

The best constant C in the definition of the space $E(a)$ (see Theorem 1.1) is a norm on $E(a)$. We won't introduce notation for it, but will refer to it in context. We find it convenient to reserve the norm symbol for an L^2 type norm to be defined later.

Let $H = -\frac{\partial^2}{\partial x^2} + x^2$ denote the harmonic oscillator. Let $\psi_t(x)$ be a solution of the Schrödinger equation

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = H\psi. \tag{3}$$

Theorem 3.1. *If $\psi_0 \in E(\tanh 2\beta)$ and $\gamma < \beta$ then for all $t \in \mathbb{R}$*

$$\psi_t \in E(\tanh \gamma),$$

with bounded norm.

The following proof was inspired by the proof of [7, Theorem 9].

Proof. Assume $\psi_0 \in E(\tanh 2\beta)$ and $\gamma < \beta$. Choose $\gamma' \in (\gamma, \beta)$ and put $r = \gamma/\gamma'$. Then $r \in (0, 1)$. The hypothesis and inequality (2) imply that

$$\langle \psi_0, \varphi_k \rangle = O(e^{-\gamma' k}).$$

If we write $\psi_0 = \sum_{n=0}^{\infty} \langle \psi_0, \varphi_n \rangle \varphi_n$, then

$$\psi_t = \sum_{n=0}^{\infty} e^{(2n+1)it} \langle \psi_0, \varphi_n \rangle \varphi_n.$$

By the Cauchy-Schwarz inequality, and Mehler's formula

$$\begin{aligned}
|\psi_t(x)| &\leq \left(\sum_{n=0}^{\infty} |\langle \psi_0, \varphi_n\rangle|^{2(1-r)}\right)^{1/2} \left(\sum_{n=0}^{\infty} |\langle \psi_0, \varphi_n\rangle|^{2r} |\varphi_n(x)|^2\right)^{1/2} \\
&\leq \frac{1}{1-e^{-2(\gamma'-\gamma)}} \left(\sum_{n=0}^{\infty} e^{-2\gamma n} |\varphi_n(x)|^2\right)^{1/2} \\
&= C(\gamma,\gamma') e^{-\frac{\tanh\gamma}{2}x^2}.
\end{aligned}$$

Also,

$$\left|\widehat{\psi_t}(x)\right| = \left|\psi_{(t-\pi/4)}(x)\right| \leq C(\gamma,\gamma') e^{-\frac{\tanh\gamma}{2}x^2}.$$

So $\psi_t \in E(\tanh\gamma)$. □

We interpret Theorem 3.1 as a result on the confinement of particle-like states of the harmonic oscillator. Regard the space $E(a)$ (strictly speaking its image in projective space) as a "Gaussian phase-box" of side $1/a$. If a state ψ_0 is initially in the phase-box of side $\coth(2\beta)$ then its evolution ψ_t is confined to the larger phase-box of side $\coth(\beta-\varepsilon)$.

The following conjecture and example show that Theorem 3.1 is almost sharp.

Conjecture 3.2. *If $\psi_0 \in E(\tanh 2\beta)$ then for all $t \in \mathbb{R}$*

$$\psi_t \in E(\tanh\beta).$$

The following example shows that we cannot do better.

Example 3.3. *Choose a branch $\sqrt{\ }$ of the square root that is defined on the right half plane and is positive on the positive real line. For $\beta > 0$, let $r = e^{-2\beta}$, and*

$$\psi_{(t-\frac{\pi}{8})} = \frac{e^{it}}{\sqrt{1+re^{4it}}} \exp\left(-\frac{1-re^{4it}}{1+re^{4it}}\frac{x^2}{2}\right)$$

Then ψ is a solution of (3),

$$\begin{aligned}
|\psi_0| &= C_0 g_{\tanh(2\beta)} \\
\left|\widehat{\psi_0}\right| = \left|\psi_{-\frac{\pi}{4}}\right| &= C_{\frac{\pi}{4}} g_{\tanh(2\beta)}, \quad \textit{but} \\
\left|\psi_{-\frac{\pi}{8}}\right| &= C_{\frac{\pi}{8}} g_{\tanh(\beta)}.
\end{aligned}$$

4. Confinement implies exponential decay

In this section, we will show that the Hermite coefficients of a "bound state" decay exponentially. We start with a simple estimate for factorials that is slightly stronger than what can be obtained from the standard Stirling formula.

Lemma 4.1. *If $\beta > 1$ then there exists $B_\beta > 0$ such that*

$$2^{-2n}\frac{(2n)!}{(n!)^2} \geq B_\beta n^{-\beta/2}, \quad n = 1, 2, \ldots.$$

Proof. Clearly, we need to prove this only for large n. Note that there exists $\delta > 0$ such that $0 \leq x \leq \delta$ implies $\log(1-x) \geq -\beta x$. Choose m so large that $k > m$ implies $0 \leq \frac{1}{2k} \leq \delta$. Put

$$D_\beta = \sum_{k=1}^{m} \log\left(1 - \frac{1}{2k}\right), \quad B_\beta = e^{D_\beta} m^{\beta/2}.$$

Let

$$Q_n = 2^{-2n}\frac{(2n)!}{(n!)^2}.$$

Then

$$\begin{aligned}\log Q_n &\geq D_\beta - \frac{\beta}{2}\sum_{k=m+1}^{n}\frac{1}{k} \\ &\geq D_\beta - \frac{\beta}{2}(\log n - \log m).\end{aligned}$$

The result follows by exponentiation. □

The results are most natural in an L^2 setting. So we define $E^2(a)$ to be the Hilbert space of all functions f such that

$$2\|f\|_a^2 = \int |f(x)|^2 e^{ax^2}\frac{dx}{\sqrt{2\pi}} + \int \left|\hat{f}(\xi)\right|^2 e^{a\xi^2}\frac{d\xi}{\sqrt{2\pi}} < \infty$$

Observe that $a_1 < a_2$ implies $E(a_2) \subseteq E^2(a_1)$.

Theorem 4.2. *For all $\alpha > 1/2$, there exists $A_\alpha > 0$ such that if $a \in (0,1)$, and ψ_t is a solution of (3) with $\|\psi_t\|_a < C$ for all $t \in \mathbb{R}$ then*

$$|\langle\psi_0, \varphi_k\rangle| \leq (C/A_\alpha)k^{\alpha/2}\left(\frac{1-a}{1+a}\right)^{k/2}$$

for $k = 1, 2, \ldots$.

Proof. Let $f = \psi_0$, and for $n \in \mathbb{Z}$, let

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} \psi_t \overline{e^{int}}\, dt.$$

Then

$$\|f_n\|_a \leq C.$$

Since f_n is an eigenfunction of the harmonic oscillator with eigenvalue n, we have $f_n = 0$ if n is even or negative, and

$$f_{2k+1} = \langle f, \varphi_k \rangle \varphi_k, \quad k = 0, 1, 2, \ldots.$$

We will get a lower bound on $\|\varphi_k\|_a$. This will imply an upper bound on $|\langle f, \varphi_k \rangle|$.

From Mehler's formula, we have

$$\sum_{k=0}^{\infty} (\varphi_k(x))^2 w^k = \sqrt{2}(1 - w^2)^{-1/2} e^{-\frac{1-w}{1+w}x^2}.$$

Multiplying both sides by e^{ax^2}, integrating, and observing that φ_k are real and are their own Fourier transforms, up to phase, we obtain

$$\sum_{k=0}^{\infty} \|\varphi_k\|_a^2 w^k = (1 - a)^{-1/2}(1 - w)^{-1/2}(1 - w/\mu)^{-1/2},$$

where $\mu = \frac{1-a}{1+a}$. Expanding the right hand side in powers of w, and equating coefficients, we obtain

$$\|\varphi_n\|_a^2 = (1 - a)^{-1/2} 2^{-2n} \sum_{k=0}^{n} \frac{(2k)!(2(n - k))!}{(k!(n - k)!)^2} \mu^{-k}$$

Since the above sum has non-negative terms, we must have

$$\|\varphi_n\|_a^2 \geq (1 - a)^{-1/2} 2^{-2n} \frac{(2n)!}{(n!)^2} \mu^{-n}.$$

So by Lemma 4.1 if $\alpha > 1/2$ there exists a constant $A_\alpha > 0$ such that

$$\|\varphi_k\|_a \geq A_\alpha (1 - a)^{-1/4} k^{-\alpha/2} \mu^{-k/2}, \quad k = 1, 2, \ldots,$$

and so

$$|\langle f, \varphi_k \rangle| \leq (C/A_\alpha)(1 - a)^{1/4} k^{\alpha/2} \mu^{k/2} = (C/A_\alpha)(1 - a)^{1/4} k^{\alpha/2} \left(\frac{1 - a}{1 + a}\right)^{k/2}, \quad k = 1, 2, \ldots.$$

□

Theorem 4.2 suggests a new approach to proving Hardy's theorem. Using PDE methods, we first prove

Theorem 4.3 (Weak confinement)**.** *There exist N such that for all $\beta > 0$, if ψ_t is a solution of (3) and $\psi_0 \in E^2(\tanh(N\beta))$ then there exists K such that*

$$\|\psi_t\|_{\tanh \beta} \leq K \|\psi_0\|_{\tanh(N\beta)}$$

for all $t \in \mathbb{R}$.

Write $a = \tanh\beta$ and $b = \tanh(N\beta)$. If $\psi_0 \in E(1)$ with norm bounded by 1, then

$$\|\psi_0\|_b \leq 2^{-1/4}(1-b)^{-1/4}$$

for all $\beta > 0$. So by Theorem 4.3, there exists K such that

$$\|\psi_t\|_a \leq K(1-b)^{-1/4}$$

So by Theorem 4.2, there exists $A > 0$ such that for all $\beta > 0$ we have

$$\begin{aligned} |\langle\psi_0, \varphi_k\rangle| &\leq \frac{K(1-b)^{-1/4}}{A}(1-a)^{1/4}k\left(\frac{1-a}{1+a}\right)^{k/2} \\ &\leq \frac{Kk}{A}e^{\frac{(N-1)\beta}{2}}e^{-\beta k} \\ &= \frac{Kk}{A}e^{\beta\left(\frac{N-1}{2}-k\right)} \quad k = 1, 2, \ldots \end{aligned}$$

It follows that ψ_0 is a *finite* linear combination of Hermite functions. Since $\psi_0 \in E(1)$, it follows that the corresponding linear combination of Hermite polynomials is bounded, and hence constant. So ψ_0 is a Gaussian.

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